Electromagnetic Field Theory (EMT)

Lecture # 22

1) Methods of Images
2) Magneto-statics
   1) Biot-Savarat’s Law
   2) Ampere’s Law
Method of Images

- The method of images, is commonly used to determine $V$, $E$, $D$, and $\rho_s$ due to charges in the presence of conductors.

- By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential.

- This method does not apply to all electrostatic problems.

- The image theory states that the field due to a charge above a perfectly conducting plane will remain the same if the conducting plane is removed and an opposite charge is placed at a symmetrical location below the plane.
Method of Images

- Examples of point, line, and volume charge configurations are shown below.
Consider a point charge $Q$ placed at a distance $h$ from a perfect conducting plane of infinite extent as shown in Figure below.
A Point Charge Above a Grounded Conducting Plane

The image configuration is in Figure below.

![Diagram of a point charge above a grounded conducting plane](image)
A Point Charge Above a Grounded Conducting Plane

The electric field at point \( P(x, y, z) \) is given by:

\[
E = E_+ + E_-
\]

\[
= \frac{Q \mathbf{r}_1}{4\pi\varepsilon_0 r_1^3} + \frac{-Q \mathbf{r}_2}{4\pi\varepsilon_0 r_2^3}
\]

The distance vectors are given as:

\[
\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h)
\]

\[
\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h)
\]

Therefore:

\[
E = \frac{Q}{4\pi\varepsilon_0} \left[ \frac{xa_x + ya_y + (z - h)a_z}{[x^2 + y^2 + (z - h)^2]^{3/2}} - \frac{xa_x + ya_y + (z + h)a_z}{[x^2 + y^2 + (z + h)^2]^{3/2}} \right]
\]
It should be noted that when \( z = 0 \), \( \mathbf{E} \) has only the \( z \)-component, confirming that \( \mathbf{E} \) is normal to the conducting surface.

The potential at \( P \) can be written as:

\[
V = V_+ + V_-
\]

\[
= \frac{Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}
\]

\[
V = \frac{Q}{4\pi\varepsilon_0} \left\{ \frac{1}{[x^2 + y^2 + (z - h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z + h)^2]^{1/2}} \right\}
\]

For \( z \geq 0 \)

and \( V = 0 \) for \( z \leq 0 \). Note that \( V(z = 0) = 0 \).
A Point Charge Above a Grounded Conducting Plane

- The surface charge density of the induced charge can be obtained as:

\[ \rho_S = D_n = \varepsilon_0 E_n \bigg|_{z=0} = -\frac{Q_h}{2\pi[x^2 + y^2 + h^2]^{3/2}}. \]

- So the total induced charge on the conducting plane is:

\[ Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-Q_h \, dx \, dy}{2\pi[x^2 + y^2 + h^2]^{3/2}}. \]

- By changing variables, \( \rho^2 = x^2 + y^2 \), \( dx \, dy = \rho \, d\rho \, d\phi \).
A Point Charge Above a Grounded Conducting Plane

Therefore:

\[ Q_i = -\frac{Qh}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\rho \, d\rho \, d\phi}{[\rho^2 + h^2]^{3/2}} \]

Or:

\[ Q_i = -\frac{Qh}{2\pi} \int_0^\infty \left[ \frac{1}{2} \right] d(\rho^2) \]

\[ = \frac{Qh}{[\rho^2 + h^2]^{1/2}} \bigg|_0^\infty \]

\[ = -Q \]

As expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.
Consider an infinite charge with density $\rho_L$ C/m located at a distance $h$ from the grounded conducting plane $z = 0$.

The same image system of point charge applies to the line charge as well except that $Q$ is replaced by $\rho_L$.

The infinite line charge $\rho_L$ may be assumed to be at $x = 0$, $z = h$ and the image $-\rho_L$ at $x = 0$, $z = -h$ so that the two are parallel to the y-axis.

The electric field at point $P$ is given as:

$$ E = E_+ + E_- $$

$$ = \frac{\rho_L}{2\pi\varepsilon_0 \rho_1} \mathbf{a}_\rho^1 + \frac{-\rho_L}{2\pi\varepsilon_0 \rho_2} \mathbf{a}_\rho^2 $$
The above equation reduces to the equation below:

\[ E = \frac{\rho L}{2\pi \varepsilon_0 \rho} a_\rho \]

Note: \( \rho \) is the perpendicular distance from the line to the point of interest.
The distance vectors are given as:

\[ \rho_1 = (x, y, z) - (0, y, h) = (x, 0, z - h) \]

\[ \rho_2 = (x, y, z) - (0, y, -h) = (x, 0, z + h) \]

So we get:

\[ E = \frac{\rho_L}{2\pi \varepsilon_0} \left[ \frac{xa_x + (z - h)a_z}{x^2 + (z - h)^2} - \frac{xa_x + (z + h)a_z}{x^2 + (z + h)^2} \right] \]

Notice that when \( z = 0 \), \( E \) has only the \( z \)-component, confirming that \( E \) is normal to the conducting surface.
A LINE Charge Above a Grounded Conducting Plane

The potential at $P$ is obtained from the line charges as:

$$V = V_+ + V_- = -\frac{\rho_L}{2\pi\varepsilon_0} \ln \rho_1 - \frac{-\rho_L}{2\pi\varepsilon_0} \ln \rho_2$$

Substituting the magnitudes of the distance vectors, we get:

$$V = -\frac{\rho_L}{2\pi\varepsilon_0} \ln \left[ \frac{x^2 + (z - h)^2}{x^2 + (z + h)^2} \right]^{1/2}$$
The surface charge induced on the conducting plane is given by:

\[ \rho_s = D_n = \varepsilon_0 E_z \bigg|_{z=0} = \frac{-\rho_L h}{\pi (x^2 + h^2)} \]

The induced charge per length on the conducting plane is:

\[ \rho_i = \int \rho_s \, dx = -\frac{\rho_L h}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x^2 + h^2} \]

By letting \( x = h \tan \alpha \), the above equation becomes:

\[ \rho_i = -\frac{\rho_L h}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{h} \]

\[ = -\rho_L \]
An infinite line charge with charge density $\rho_L = \rho_0$ lies on the $z$-axis. Two infinite conducting planes are located at $y = a$ and $y = a - h$ and both have zero potential. 

Find the voltage at any given point $(x, y)$. If $\rho_0 = 1.0 \times 10^{-7}$ C/m, $a = 1.0$ m and $h = 2.0$ m, plot the contours of the voltage.
A positive point charge $Q$ is located at distance $d_1$ and $d_2$ respectively from two grounded perpendicular conducting half planes. Determine the force on charge $Q$ caused by the charges induced on the planes.
Let surface $y=0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x=0$, $y=1$ and $x=0$, $y=2$. Let $V=0$ at the plane $y=0$, find $\mathbf{E}$ at $P(1,2,0)$.
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1) Methods of Images
2) Magneto-statics
   1) Biot-Savarat’s Law
   2) Ampere’s Law
We have studied that an electrostatic field is produced by static or stationary charges.

If the charges are moving with constant velocity, a static magnetic (or magneto-static) field is produced.

A magneto-static field is produced by a constant current flow (or direct current).

This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires.

Applications: motors, transformers, microphones, compasses ….
Two Major Laws in Magneto-static

- **Biot-Savart’s Law**
  Like Coulomb's law, Biot-Savart's law is the general law of magneto-statics.

- **Ampere’s Law**
  - Like Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot-Savart's law.
  - Ampere’s law is easily applied in problems involving symmetrical current distribution.
Biot Savart Law

The contribution to the magnetic field \((dH)\) at a point \(P\) is directly proportional to:

1. The current \(I\) flowing through the wire,
2. The differential length \(dl\),
3. The sine of the angle between the differential length and the direction to the observation point \(\alpha\).

Inversely proportional to:

1. The square of the distance between the current element and the observation point \(R\).
The cross product could be viewed "how much a vector is perpendicular to another vector".

If two vectors have the same direction (or have the exact opposite direction from one another) or if either one has zero length, then their cross product is zero.
Biot Savart Law - Mathematical Form

\[ dH \propto \frac{I \, dl \, \sin \alpha}{R^2} \]

or

\[ dH = \frac{kI \, dl \, \sin \alpha}{R^2} \]

where \( k \) is the constant of proportionality

In SI units, \( k = \frac{1}{4\pi} \), so:

\[ dH = \frac{I \, dl \, \sin \alpha}{4\pi R^2} \]

From the definition of cross product:

\[ dH = \frac{I \, dl \times a_R}{4\pi R^2} = \frac{I \, dl \times \mathbf{R}}{4\pi R^3} \]

where \( R = |\mathbf{R}| \) and \( a_R = \mathbf{R}/R \).
The direction of $dH$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $dH$. 

$\text{Image of right-hand rule and wire}$
Direction of $dH$

$H$ (or $I$) is out

$H$ (or $I$) is in

$dl$ at angle $\alpha$

$I$

$P$

$dH$
Distributed Current Sources

- Just as we can have different charge configurations, we can have different current distributions:
  1. Line current,
  2. Surface current, and

- The source elements are related as:

\[ I \, dl = K \, dS = J \, dv \]

- \( K \) is the **surface current density** (amperes/meter)

- \( J \) is the **volume current density** (amperes/meter square)
Distributed Current Sources

\[ H = \int_L \frac{I \, dl \times a_R}{4\pi R^2} \]

\[ H = \int_S \frac{K \, dS \times a_R}{4\pi R^2} \]

\[ H = \int_v \frac{J \, dv \times a_R}{4\pi R^2} \]
As an example, let us apply Biot-Savart law to determine the field due to a straight current carrying filamentary conductor of finite length AB.

We assume that the conductor is along the z-axis with its upper and lower ends respectively subtending angles $\alpha_2$ and $\alpha_1$ at $P$, the point at which $\mathbf{H}$ is to be determined.

Particular note should be taken of this assumption as the formula to be derived will have to be applied accordingly.
We consider the contribution $d\mathbf{H}$ at $P$ due to an element $dl$ at $(0, 0, z)$

$$d\mathbf{H} = \frac{I \, dl \times \mathbf{R}}{4\pi R^3}$$

Since $dl = dz \mathbf{a}_z$ and $\mathbf{R} = \rho \mathbf{a}_\rho - z \mathbf{a}_z$, so:

$$dl \times \mathbf{R} = \rho \, dz \, \mathbf{a}_\phi$$

Hence:

$$\mathbf{H} = \int \frac{I \rho \, dz}{4\pi [\rho^2 + z^2]^{3/2}} \mathbf{a}_\phi$$

Let $z = \rho \cot \alpha$, $dz = -\rho \csc^2 \alpha \, d\alpha$, we get:

$$\mathbf{H} = -\frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \csc^2 \alpha \, d\alpha}{\rho^3 \csc^3 \alpha} \mathbf{a}_\phi$$

$$= -\frac{I}{4\pi \rho} \mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin \alpha \, d\alpha$$
Current Carrying Filament

- Or:

\[
H = \frac{I}{4\pi \rho} (\cos \alpha_2 - \cos \alpha_1) a_\phi
\]

- This expression is generally applicable for any straight filamentary conductor of finite length.

- Notice that \(H\) is always along the unit vector \(a_\phi\) (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest \(P\).

- As a special case, when the conductor is semi-infinite (with respect to \(P\)) so that point \(A\) is now at \((0, 0, 0)\) while \(B\) is at \((0, 0, \infty)\); \(\alpha_1 = 90^\circ\), \(\alpha_2 = 0^\circ\), the above becomes:

\[
H = \frac{I}{4\pi \rho} a_\phi
\]
Another special case is when the conductor is infinite in length.

For this case, point A is at \((0, 0, -\infty)\) while B is at \((0, 0, \infty)\); \(\alpha_1 = 180^\circ\), \(\alpha_2 = 0^\circ\), so the equation reduces to:

\[
\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi
\]

A simple method to determine the unit vector \(\mathbf{a}_\phi\) is to use the relation below:

\[
\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho
\]

Here \(\mathbf{a}_\ell\) is a unit vector along the line current and \(\mathbf{a}_\rho\) is a unit vector along the perpendicular line from the line current to the field point.
The conducting triangular loop in the figure carries a current of 10 A.

a) Find $\mathbf{H}$ at (0, 0, 5) due to side 1 of the loop.
Solution-1

\[ \cos \alpha_1 = \cos 90^\circ = 0, \]
\[ \cos \alpha_2 = \frac{2}{\sqrt{29}}, \quad \rho = 5 \]
\[ \mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y \]

\[ \mathbf{H} = \frac{I}{4\pi \rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi \]
\[ = \frac{10}{4\pi (5)} \left( \frac{2}{\sqrt{29}} - 0 \right)(-\mathbf{a}_y) \]
\[ = -59.1 \mathbf{a}_y \text{ mA/m} \]
A circular loop located on $x^2 + y^2 = 9, z = 0$ carries a direct current of 10 A along $a_\phi$. Determine:

a) $\mathbf{H}$ at P(0, 0, h)

b) $\mathbf{H}$ at (0,0,4) and (0, 0, -4).
\[ d\mathbf{H} = \frac{I \, dl \times \mathbf{R}}{4\pi R^3} \]

\[ dl = \rho \, d\phi \, \mathbf{a}_\phi \]

\[ \mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho \mathbf{a}_\rho + h \mathbf{a}_z, \]

\[ dl \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho \, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h \, d\phi \, \mathbf{a}_\rho + \rho^2 \, d\phi \, \mathbf{a}_z \]
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Ampere’s Circuit Law

- Ampere's circuit law states that “the line integral of the tangential component of \( \mathbf{H} \) around a closed path is the same as the net current \( I_{\text{enc}} \) enclosed by the path”

\[
\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}
\]

- The closed integral in the above expression can be performed on any closed path “a” or “b” or “c”
Maxwell’s Third Equation

We have the following equation from Ampere’s law:

\[ \oint H \cdot dl = I_{enc} \]

Applying Stoke’s Theorem to the left-hand side, we get:

\[ I_{enc} = \oint H \cdot dl = \int (\nabla \times H) \cdot dS \]

But

\[ I_{enc} = \int J \cdot dS \]

Comparing the two equations above, we get

\[ \nabla \times H = J \]

This is Maxwell’s Third Equation also called Ampere’s Law in point or differential form.